Description of random fields by means of one-point finite-conditional distributions

Arnak S. Dalalyan
Imagine/LIGM, Ecole des Ponts ParisTech, Université Paris-Est, FRANCE

Boris S. Nahapetian
Institute of Mathematics, National Academy of Sciences, ARMENIA

(Dated: November 12, 2010)

The aim of this note is to investigate the relationship between strictly positive random fields on a lattice $\mathbb{Z}^\nu$ and the conditional probability measures at one point given the values on a finite subset of the lattice $\mathbb{Z}^\nu$. We exhibit necessary and sufficient conditions for a one-point finite-conditional system to correspond to a unique strictly positive probability measure. It is noteworthy that the construction of the aforementioned probability measure is done explicitly by some simple procedure. Finally, we introduce a condition on the one-point finite conditional system that is sufficient for ensuring the mixing of the underlying random field.

Keywords: Random field, one-point conditional distribution, mixing properties

I. INTRODUCTION

The mathematical theory of random fields is an active area of research studying the probabilistic properties of systems of interacting particles. In recent years, random fields have been successfully applied to the analysis of biological sequences, text and image processing, as well as to many areas of computer vision and artificial intelligence. In most of these applications, a random field is defined by its finite dimensional conditional distributions and is therefore often termed conditional random field. The reconstruction of the distributions of random fields from these conditional probabilities is the subject of this paper.

The description of a random field by means of its conditional distributions is an old problem, most important contributions to which date back to Dobrushin [5, 6]. In his seminal paper [6], Dobrushin considered systems of conditional distributions on finite sets under the condition that the values of the field are known outside this set and proved that, under some assumptions, there exists a random field with the given conditional distributions. This line of research has been further developed in recent papers [1–4, 8]. The aim of this note is to complement these works by providing an exhaustive description of one-point finite-conditional distributions that give rise to a positive random field.

To be more precise, we consider a random field $\boldsymbol{X}$ on the ν-dimensional regular grid $\mathbb{Z}^\nu$ and with values in a finite set $\mathscr{X}$. Given the distribution of $\boldsymbol{X}$, the conditional probabilities $\boldsymbol{Q}_t^{X_\Lambda}(x) = \mathbf{P}(X_t = x|X_\Lambda)$ can be easily computed for every $x \in \mathscr{X}$ and for every finite set $\Lambda \subset \mathbb{Z}^\nu$. In some situations, however, the random field may be unavailable and only a set of conditional distributions $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$ can be defined. In image segmentation, for instance, it is more convenient [9, 11] to define a random field $\boldsymbol{X}$ by specifying the conditional distribution of $\boldsymbol{X}$ at any lattice point t given its values on the neighboring points. The segmentation is then obtained by assigning to each point t the most likely value taken by X_t (cf. Figure 1 for an example). In such a situation, it is relevant to raise the question of the existence of a random field corresponding to a set of conditional distributions $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$. This is the main issue studied in this work; we prove that under some consistency assumptions on the collection $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$, there exists a random field corresponding to this collection. Furthermore, the distribution of this random field is uniquely determined by the collection $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$.

FIG. 1: A natural image and its segmentation into 4 regions. The segmented image is obtained as the most likely configuration w.r.t. a probability measure corresponding to a random field on $\mathbb{Z}^2$ with state space $\{1, 2, 3, 4\}$.

The rest of the paper is organized as follows. In Section II, we introduce the main notation used throughout the paper and present the mathematical formulation of the questions we are interested in. The main result of the paper is stated and proved in Section III. We briefly discuss the mixing properties in Section IV and summarize the main findings of the paper in Section V.

II. NOTATION AND PROBLEM FORMULATION

Let $\boldsymbol{X}$ be a random field on $\mathbb{Z}^\nu$ with a finite state space $\mathscr{X}$ drawn from a probability distribution $\mathbf{P}$ on $(\mathscr{X}^{\mathbb{Z}^\nu}, \mathscr{A})$, where the σ-algebra $\mathscr{A}$ is defined as $(2^{\mathscr{X}})^{\mathbb{Z}^\nu}$ with $2^{\mathscr{X}}$ being the set of all subsets of $\mathscr{X}$. Note that $\mathbf{P}$ is a probability measure acting on an infinite-dimensional space. A classical way of characterizing such a probability measure passes through the collection of its finite-dimensional distributions : $\big\{\mathbf{P}_\Lambda,\ \Lambda \subset \mathbb{Z}^\nu \text{ and } \mathrm{Card}(\Lambda) < \infty\big\}$. The famous result of Kolmogorov states that a collection of finite-dimensional distributions correspond to a unique probability measure on $(\mathscr{X}^{\mathbb{Z}^\nu}, \mathscr{A})$ if and only if it satisfies Kolmogorov's consistency condition.

In this work, we will focus our attention on strictly positive random fields, *i.e.*, random fields $\boldsymbol{X}$ satisfying $\mathbf{P}(X_\Lambda = x_\Lambda) > 0$ for all non-empty, finite sets $\Lambda \subset \mathbb{Z}^\nu$ and for all $x_\Lambda \in \mathscr{X}^\Lambda$. For such a random field, the one-point finite-conditional probabilities are defined as follows. For any $\Lambda \subset \mathbb{Z}^\nu$, let us denote by $\widetilde{\mathscr{X}^\Lambda}$ the set of all functions $\widetilde{\boldsymbol{x}}$ defined on some non-empty, finite subset J of $\mathbb{Z}^\nu \setminus \Lambda$ and taking values in $\mathscr{X}$: $\widetilde{\boldsymbol{x}} : J \to \mathscr{X}$. We will refer to J as the support of $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^\Lambda}$. For every $t \in \mathbb{Z}^\nu$ and for every $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}$, we can define the conditional probability measure on $\mathscr{X}$:

$$\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(\cdot) = \mathbf{P}(X_t = \cdot \,|\, X_J = \widetilde{\boldsymbol{x}}), \qquad \text{where } J = \mathrm{supp}(\widetilde{\boldsymbol{x}}).$$

The set $\boldsymbol{q}(\mathbf{P}) = \big\{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}\big\}$ is called one-point finite-conditional distribution of $\mathbf{P}$. The problems we are interested in can be formulated as those related to the inversion of the operator $\boldsymbol{q}$.

To be more precise, let $\widetilde{\boldsymbol{q}} = \{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}\}$ be a system of probability distributions on $\mathscr{X}$ such that $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) > 0$ for all $x \in \mathscr{X}$. We define $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{q}})$ the set of all strictly positive probability measures $\mathbf{P}$ such that $\boldsymbol{q}(\mathbf{P}) = \widetilde{\boldsymbol{q}}$; in other terms $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{q}}) = \boldsymbol{q}^{-1}(\widetilde{\boldsymbol{q}})$. The main goal of this work is to accomplish the following tasks:

(a) Determine necessary and sufficient conditions on $\widetilde{\boldsymbol{q}}$ guaranteeing that the set $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{q}})$ is non-empty.

(b) Prove that if $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{q}})$ is non-empty then it is a singleton, that is there is a unique random field having $\widetilde{\boldsymbol{q}}$ as a one-point finite-conditional distribution.

(c) Exhibit some conditions on $\widetilde{\boldsymbol{q}}$ entailing that the corresponding random field, if exists, enjoys mixing properties.

III. NECESSARY AND SUFFICIENT CONDITION FOR EXISTENCE AND UNIQUENESS

It is quite clear that not every one-point finite-conditional distribution $\widetilde{\boldsymbol{q}}$ corresponds to a random field. For instance, it is obvious that for a random field with strictly positive probability distribution the following property

$$\mathbf{P}\big((X_t, X_s) = (x,y) \,|\, X_J = \widetilde{\boldsymbol{x}}\big) = \mathbf{P}\big(X_t = x \,|\, X_J = \widetilde{\boldsymbol{x}}\big)\mathbf{P}\big(X_s = y \,|\, X_J = \widetilde{\boldsymbol{x}}, X_t = x\big) \tag{1}$$

should be satisfied for every $t, s \in \mathbb{Z}^\nu$, $(x,y) \in \mathscr{X}^{\{t,s\}}$ and for all $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^{\{t,s\}}}$. Therefore, if for $\widetilde{\boldsymbol{q}}$ the condition $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x)\boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}x}(y) = \boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}}(y)\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}y}(x)$ fails for some $(s,t,x,y,\widetilde{\boldsymbol{x}})$, then there is no random field having $\widetilde{\boldsymbol{q}}$ as the one-point finite-conditional distribution. The next theorem provides a precise characterization of systems $\widetilde{\boldsymbol{q}}$ that can be extended to a strictly positive random field. Furthermore, it shows that the corresponding random field is unique and can be constructed by a simple procedure.

Theorem 1. *Let $\widetilde{\boldsymbol{q}} = \{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu$ and $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}\}$ be a one-dimensional finite-conditional distribution on $\mathscr{X}$. There exists a strictly positive random field having $\widetilde{\boldsymbol{q}}$ as system of conditional probabilities, i.e., $\boldsymbol{\mathcal{P}}(\widetilde{\boldsymbol{q}}) \neq \varnothing$, if and only if the following conditions are fulfilled:*

[C1] *For all $t \in \mathbb{Z}^\nu$, $x \in \mathscr{X}$ and $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}$, it holds that $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) > 0$.*

[C2] *For all $t, s \in Z^\nu$, $x \in \mathscr{X}$, $y \in \mathscr{X}$ and $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^{\{t,s\}}}$, it holds that*

$$\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x)\boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}x}(y) = \boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}}(y)\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}y}(x).$$

[C3] *For all $t, s \in Z^\nu$, $x, x' \in \mathscr{X}^t$ and $y, y' \in \mathscr{X}^s$, it holds that*

$$\boldsymbol{Q}_t^{y}(x)\boldsymbol{Q}_s^{x'}(y)\boldsymbol{Q}_t^{y'}(x')\boldsymbol{Q}_s^{x}(y') = \boldsymbol{Q}_t^{y'}(x)\boldsymbol{Q}_s^{x'}(y')\boldsymbol{Q}_t^{y}(x')\boldsymbol{Q}_s^{x}(y). \tag{2}$$

Furthermore, under these conditions, all the random fields having $\widetilde{\boldsymbol{q}}$ as conditional distributions have the same distribution.

Proof. Let us start by proving that all three conditions stated in the theorem are necessary. This is obviously the case for the first condition. For the second one, the necessity follows from the property of random fields stated in Eq. (1). Finally,let us remark that if P is a probability measure and A, B, C, D are some arbitrary events, then

$$P(A|B)P(B|C)P(C|D)P(D|A) = P(A|D)P(D|C)P(C|B)P(B|A). \tag{3}$$

Applying this identity to the random field $\boldsymbol{X}$ drawn from $\mathbf{P}$ and to the events $A = \{X_t = x\}$, $B = \{X_s = y\}$, $C = \{X_t = x'\}$ and $D = \{X_s = y'\}$ we get the necessity of condition (2).

To prove that these three conditions are sufficient, we assume that $\widetilde{\boldsymbol{q}}$ is a one-point finite-conditional system satisfying conditions [C1-C3]. For any $t \in \mathbb{Z}^\nu$, we choose some $s \in \mathbb{Z}^\nu \setminus \{t\}$ and some $y \in \mathscr{X}^s$, and set

$$\mathbf{P}_t(x) = \frac{\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_s^x(y)} \left[\sum_{u \in \mathscr{X}^t} \frac{\boldsymbol{Q}_t^y(u)}{\boldsymbol{Q}_s^u(y)} \right]^{-1}, \qquad \forall x \in \mathscr{X}^t. \tag{4}$$

Let us show that under condition [C2] and [C3] $P_t(x)$ defined as above is independent of s and of y. Indeed, it follows from [C2] that, for every distinct points $t, s, r \in \mathbb{Z}^\nu$ and for every $x \in \mathscr{X}^t$, $y \in \mathscr{X}^s$, $z \in \mathscr{X}^r$,

$$\begin{aligned}
\boldsymbol{Q}_t^z(x)\boldsymbol{Q}_s^{zx}(y) &= \boldsymbol{Q}_s^z(y)\boldsymbol{Q}_t^{zy}(x),\\
\boldsymbol{Q}_r^y(z)\boldsymbol{Q}_t^{zy}(x) &= \boldsymbol{Q}_t^y(x)\boldsymbol{Q}_r^{yx}(z),\\
\boldsymbol{Q}_s^x(y)\boldsymbol{Q}_r^{yx}(z) &= \boldsymbol{Q}_r^x(z)\boldsymbol{Q}_s^{zx}(y).
\end{aligned}$$

Multiplying these equations, we see that many terms cancel out and we end up with the identity

$$\boldsymbol{Q}_t^z(x)\boldsymbol{Q}_s^x(y)\boldsymbol{Q}_r^y(z) = \boldsymbol{Q}_s^z(y)\boldsymbol{Q}_r^x(z)\boldsymbol{Q}_t^y(x). \tag{5}$$

Since (5) holds for every $x \in \mathscr{X}^t$, we have also

$$\boldsymbol{Q}_t^z(u)\boldsymbol{Q}_s^u(y)\boldsymbol{Q}_r^y(z) = \boldsymbol{Q}_s^z(y)\boldsymbol{Q}_r^u(z)\boldsymbol{Q}_t^y(u), \qquad \forall u \in \mathscr{X}^t. \tag{6}$$

Dividing (5) by (6), we get

$$\frac{\boldsymbol{Q}_t^z(x)\boldsymbol{Q}_s^x(y)}{\boldsymbol{Q}_t^z(u)\boldsymbol{Q}_s^u(y)} = \frac{\boldsymbol{Q}_r^x(z)\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_r^u(z)\boldsymbol{Q}_t^y(u)}. \tag{7}$$

Rearranging the terms, we come to the equality

$$\frac{\boldsymbol{Q}_t^z(x)\boldsymbol{Q}_t^y(u)}{\boldsymbol{Q}_r^x(z)\boldsymbol{Q}_s^u(y)} = \frac{\boldsymbol{Q}_t^y(x)\boldsymbol{Q}_t^z(u)}{\boldsymbol{Q}_s^x(y)\boldsymbol{Q}_r^u(z)}, \tag{8}$$

which implies after summing w.r.t. $u \in \mathscr{X}^t$ that

$$\frac{\boldsymbol{Q}_t^z(x)}{\boldsymbol{Q}_r^x(z)} \sum_u \frac{\boldsymbol{Q}_t^y(u)}{\boldsymbol{Q}_s^u(y)} = \frac{\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_s^x(y)} \sum_u \frac{\boldsymbol{Q}_t^z(u)}{\boldsymbol{Q}_r^u(z)}. \tag{9}$$

In other terms,

$$\frac{\boldsymbol{Q}_t^z(x)}{\boldsymbol{Q}_r^x(z)} \left[\sum_u \frac{\boldsymbol{Q}_t^z(u)}{\boldsymbol{Q}_r^u(z)} \right]^{-1} = \frac{\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_s^x(y)} \left[\sum_u \frac{\boldsymbol{Q}_t^y(u)}{\boldsymbol{Q}_s^u(y)} \right]^{-1}. \tag{10}$$

This equality proves that the definition of $\mathbf{P}_t$ given in (4) does not depend on the choice of s and y: if we choose $r \neq s$ instead of s and z instead of y, the result remains unchanged. The fact that the right hand side of (4) does not depend on y, follows in a direct manner from the condition [C3].

So far, we proved that given a one-point finite-conditional distribution $\widetilde{\boldsymbol{q}}$ satisfying [C1-C3], one can uniquely determine the one-point unconditional distributions. Let us now look at what happens with the remaining finite dimensional unconditional distributions. To this end, let Λ be a finite subset of $\mathbb{Z}^d$ the elements of which are somehow enumerated: $\Lambda = \{t_1, \ldots, t_n\}$. For every $\boldsymbol{x}_\Lambda \in \mathscr{X}^\Lambda$, we define

$$\mathbf{P}_\Lambda(\boldsymbol{x}_\Lambda) = \mathbf{P}_{t_1}(x_{t_1}) \boldsymbol{Q}_{t_2}^{x_{t_1}}(x_{t_2}) \cdot \ldots \cdot \boldsymbol{Q}_{t_n}^{x_{t_1} \ldots x_{t_{n-1}}}(x_{t_n}), \tag{11}$$

where $\mathbf{P}_{t_1}(x_{t_1})$ is (well) defined by (4). We prove below that this definition is independent of the enumeration of the elements of Λ and that the family $\{\mathbf{P}_\Lambda : \Lambda \text{ is a finite subset of } \mathbb{Z}^\nu\}$ is a collection of probability measures that are consistent in the Kolmogorov sense.

To prove that the definition of $\mathbf{P}_\Lambda$ is invariant w.r.t. the order on the elements of Λ, we use the fact that any permutation of $t_1, \ldots, t_n$ can be obtained as the composition of a finite number of permutations of two successive elements. Therefore, it is sufficient to prove that considering the ordered set $\{t_1, t_2, \ldots, t_k, t_{k-1}, \ldots, t_n\}$ instead of $\{t_1, t_2, \ldots, t_{k-1}, t_k, \ldots, t_n\}$ leave the definition of $\mathbf{P}_\Lambda$ unchanged. Thus, we aim at showing that

$$\boldsymbol{Q}_{t_{k-1}}^{x_{t_1} \ldots x_{t_{k-2}}}(x_{t_{k-1}}) \boldsymbol{Q}_{t_k}^{x_{t_1} \ldots x_{t_{k-1}}}(x_{t_k}) = \boldsymbol{Q}_{t_k}^{x_{t_1} \ldots x_{t_{k-2}}}(x_{t_k}) \boldsymbol{Q}_{t_{k-1}}^{x_{t_1} \ldots x_{t_{k-2}}, x_{t_k}}(x_{t_{k-1}}), \qquad \forall k > 2, \tag{12}$$

and that

$$\mathbf{P}_{t_1}(x_{t_1}) \boldsymbol{Q}_{t_2}^{x_{t_1}}(x_{t_2}) = \mathbf{P}_{t_2}(x_{t_2}) \boldsymbol{Q}_{t_1}^{x_{t_2}}(x_{t_1}) \tag{13}$$

Equation (12) reduces to condition [C2] by setting $t = t_{k-1}$, $s = t_k$, $\widetilde{\boldsymbol{x}} = \{x_{t_1}, \ldots, x_{t_{k-2}}\}$, $x = x_{t_{k-1}}$ and $y = x_{t_k}$. The case of Eq. (13) is a bit more delicate and requires the use of [C3]. To simplify notation, we set $t = t_1$, $s = t_2$, $x_{t_1} = x$ and $x_{t_2} = y$. We wish to show that $\mathbf{P}_t(x) \boldsymbol{Q}_s^x(y) = \mathbf{P}_s(y) \boldsymbol{Q}_t^y(x)$, which amounts to

$$\boldsymbol{Q}_t^y(x) \left[\sum_{x' \in \mathscr{X}^t} \frac{\boldsymbol{Q}_t^y(x')}{\boldsymbol{Q}_s^{x'}(y)} \right]^{-1} = \boldsymbol{Q}_s^x(y) \left[\sum_{y' \in \mathscr{X}^s} \frac{\boldsymbol{Q}_s^x(y')}{\boldsymbol{Q}_t^{y'}(x)} \right]^{-1}.$$

This can be equivalently written as

$$\sum_{y' \in \mathscr{X}^s} \frac{\boldsymbol{Q}_s^x(y') \boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_t^{y'}(x)} = \sum_{x' \in \mathscr{X}^t} \frac{\boldsymbol{Q}_t^y(x') \boldsymbol{Q}_s^x(y)}{\boldsymbol{Q}_s^{x'}(y)}. \tag{14}$$

Using the fact that $\sum_{x'} \boldsymbol{Q}_t^{y'}(x') = \sum_{y'} \boldsymbol{Q}_s^{x'}(y') = 1$ one can rewrite Eq. (14) as follows:

$$\sum_{x' \in \mathscr{X}^t} \sum_{y' \in \mathscr{X}^s} \frac{\boldsymbol{Q}_s^x(y') \boldsymbol{Q}_t^y(x) \boldsymbol{Q}_s^{x'}(y) \boldsymbol{Q}_t^{y'}(x')}{\boldsymbol{Q}_t^{y'}(x) \boldsymbol{Q}_s^{x'}(y)} = \sum_{y' \in \mathscr{X}^s} \sum_{x' \in \mathscr{X}^t} \frac{\boldsymbol{Q}_t^y(x') \boldsymbol{Q}_s^x(y) \boldsymbol{Q}_t^{y'}(x) \boldsymbol{Q}_s^{x'}(y')}{\boldsymbol{Q}_t^{y'}(x) \boldsymbol{Q}_s^{x'}(y)}. \tag{15}$$

Now, it follows from [C3] that the equality (15) is true. Thus, if conditions [C2] and [C3] are fulfilled, then the distribution $\mathbf{P}_\Lambda$ is the same irrespectively on the choice of the enumeration.

In order to prove the consistency in Kolmogorov's sense, we use the fact that $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(\cdot)$ is a probability measure on $\mathscr{X}$. Therefore, $\sum_{x \in \mathscr{X}^t} \boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) = 1$, implying thus that for any finite $\Lambda \subset \mathbb{Z}^\nu$, for any $t \in \mathbb{Z}^\nu \setminus \Lambda$ and for any $\widetilde{\boldsymbol{x}} \in \mathscr{X}^\Lambda$,

$$\sum_{x \in \mathscr{X}^t} \mathbf{P}_{\Lambda \cup \{t\}}(\widetilde{\boldsymbol{x}} x) = \sum_{x \in \mathscr{X}^t} \mathbf{P}_\Lambda(\widetilde{\boldsymbol{x}}) \boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) = \mathbf{P}_\Lambda(\widetilde{\boldsymbol{x}}).$$

This concludes the proof. □

IV. SUFFICIENT CONDITION FOR MIXING

It is of primary interest in probability theory to study the mixing properties [7] of random fields, since they characterize the behavior of additive functionals of the random field by means of central limit theorems [10]. The aim of this section is to describe a simple condition—on a one-point finite-conditional distribution satisfying conditions [C1-C3]—that allows to evaluate the mixing properties of the underlying random field. To this end, we introduce—for every pair of distinct point $t, s \in \mathbb{Z}^\nu$—the notation

$$\rho_{s,t} = \sup_{\substack{\Lambda: s\in\Lambda, t\notin\Lambda, |\Lambda|<\infty \\ \widetilde{\boldsymbol{x}}:\mathrm{supp}(\widetilde{\boldsymbol{x}})=\Lambda\setminus\{s\}}} \max_{\bar{x},\widetilde{x}\in\mathscr{X}^s} \sum_{x\in\mathscr{X}^t} |\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}\widetilde{x}}(x) - \boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}\bar{x}}(x)|. \tag{16}$$

Theorem 2. *Let $\widetilde{\boldsymbol{q}} = \{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu$ and $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}\}$ be a one-dimensional finite-conditional distribution on $\mathscr{X}$ satisfying conditions of Theorem 1. For every pair of disjoint finite subsets I and V of $\mathbb{Z}^\nu$, the random field $\mathbf{P}$ reconstructed from $\widetilde{\boldsymbol{q}}$ satisfies the inequality*

$$\max_{\substack{\boldsymbol{x}\in\mathscr{X}^V \\ \boldsymbol{y}\in\mathscr{X}^I}} \left|\mathbf{P}_V(\boldsymbol{x}) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right| \le \sum_{s\in I}\sum_{t\in V} \rho_{s,t}. \tag{17}$$

Proof. Let us denote by m and n the cardinalities of I and V, respectively. We will proceed by induction on $m+n$.

Assume first that $m+n=2$. This implies that $m=n=1$ and therefore $V=\{t\}$ and $I=\{s\}$. Therefore,

$$\begin{aligned}\left|\mathbf{P}_t(x) - \mathbf{P}_{t|s}(x|y)\right| &= \left|\sum_{z\in\mathscr{X}^s} \boldsymbol{Q}_t^z(x)\mathbf{P}_s(z) - \boldsymbol{Q}_t^y(x)\right| \\ &\le \sum_{z\in\mathscr{X}^s} \mathbf{P}_s(z)\left|\boldsymbol{Q}_t^z(x) - \boldsymbol{Q}_t^y(x)\right| \\ &\le \max_{z\in\mathscr{X}^s}\left|\boldsymbol{Q}_t^z(x) - \boldsymbol{Q}_t^y(x)\right| \le \rho_{s,t}.\end{aligned} \tag{18}$$

This proves the claim of the theorem in the case $m+n=2$.

Assume now that the claim of the theorem is valid for every pair of strictly positive integers (m,n) such that $m+n\le k$. Our aim is to prove (17) in the case where $m+n=k+1$. Let us arbitrarily choose a point u in I and set $J = I\setminus\{u\}$. Since the claim of the theorem is assumed to hold true for the pair (V,J), we have

$$\begin{aligned}\left|\mathbf{P}_V(\boldsymbol{x}) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right| &\le \left|\mathbf{P}_V(\boldsymbol{x}) - \mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J)\right| + \left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right| \\ &\le \sum_{s\in J}\sum_{t\in V}\rho_{s,t} + \left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right|.\end{aligned} \tag{19}$$

One easily checks that for every $t'\in V$,

$$\begin{aligned}\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) &= \boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'})\mathbf{P}_{V\setminus\{t'\}|J}(\boldsymbol{x}_{V\setminus\{t'\}}|\boldsymbol{y}_J) \\ \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y}) &= \boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'})\mathbf{P}_{V\setminus\{t'\}|I}(\boldsymbol{x}_{V\setminus\{t'\}}|\boldsymbol{y})\end{aligned}$$

and hence

$$\begin{aligned}\left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right| \le &\left|\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'}) - \boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'})\right| \\ &+ \left|\mathbf{P}_{V\setminus t'|J}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y}_J) - \mathbf{P}_{V\setminus t'|I}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y})\right|.\end{aligned} \tag{20}$$

It follows from the formula of total probabilities that

$$\begin{aligned}&\left|\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus t'}}(x_{t'})-\boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus t'}}(x_{t'})\right|\\&\qquad=\Big|\sum_{z\in\mathscr{X}^u}\Big(\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,z,\boldsymbol{x}_{V\setminus t'}}(x_{t'})-\boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus t'}}(x_{t'})\Big)\mathbf{P}(z|\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus t'})\Big|\\&\qquad\le\max_{z\in\mathscr{X}^u}\left|\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,z,\boldsymbol{x}_{V\setminus t'}}(x_{t'})-\boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus t'}}(x_{t'})\right|\le\rho_{u,t'}.\end{aligned}$$

Combining with (20), we get

$$\left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J)-\mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right|\le\rho_{u,t'}+\left|\mathbf{P}_{V\setminus t'|J}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y}_J)-\mathbf{P}_{V\setminus t'|I}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y})\right|.$$

Repeating this argument, we find that

$$\left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J)-\mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right|\le\sum_{t'\in V}\rho_{u,t'}.$$

This estimate, in conjunction with inequality (19), completes the proof of the theorem. □

As an application of this result, let us consider the one-dimensional case $\nu=1$. Assume that there exist $\rho_*<1$ and $d_*>0$ such that $\rho_{s,t}\le\rho_*^{|t-s|}$ as soon as $|t-s|\ge d_*$. In this case, one easily checks that for every pair of finite intervals $V,I\subset\mathbb{Z}$ such that $d=d(V,I)=\min_{t\in V,s\in I}|t-s|\ge d_*$, it holds that

$$\sum_{s\in I}\sum_{t\in V}\rho_{s,t}\le\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\rho_*^{d+i+j}=\sum_{k=0}^{\infty}(k+1)\rho_*^{d+k}=\frac{\rho_*^d}{(1-\rho_*)^2}\ .$$

This short computation shows that the quantity $|\mathbf{P}_V(\boldsymbol{x})-\mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})|$ decreases exponentially to zero when the distance between two intervals V and I tend to infinity.

V. CONCLUSION

In this work, we have introduced the notion of the one-point finite-conditional distributions and established necessary and sufficient conditions (cf. [C1-C3] in Theorem 1) for such a system to be the set of conditional probabilities of a strictly positive random field on $\mathbb{Z}^\nu$ and with finite state-space. Conditions [C2-C3], which are the most important ones, can be seen as consistency conditions in the same spirit as the Kolmogorov consistency conditions for finite-dimensional distributions of a random process indexed by an infinite set. We have further demonstrated that it is possible to assess the rate of mixing of a random field by evaluating some characteristics on the one-point finite-conditional distributions, without resorting to the computation of the unconditional distributions of the random field.

A relevant open question concerns the relaxation of the assumption of strong positiveness, *e.g.* by introducing a notion of weakly positive random fields in the spirit of [2].

Acknowledgments

We wish to thank Sergueï Dachian for his careful reading of the initial version of the manuscript and his numerous valuable suggestions.

---

[1] Dachian, S. and Nahapetian, B. S. Description of random fields by means of one-point conditional distributions and some applications. *Markov Process. Related Fields* **7** (2001), no. 2, 193–214.

[2] Dachian, S. and Nahapetian, B. S. Description of specifications by means of probability distributions in small volumes under condition of very weak positivity. *J. Statist. Phys.* **117** (2004), no. 1-2, 281–300.

[3] Dachian, S. and Nahapetian, B. S. On Gibbsianness of random fields. *Markov Process. Related Fields* **15** (2009), no. 1, 81–104.

[4] Dashyan, S. Yu. and Nakhapetyan, B. S. Inclusion-exclusion description of random fields. *J. Contemp. Math. Anal.* **30** (1995), no. 6, 50–61.

[5] Dobrushin, R. L. The problem of uniqueness of a Gibbsian random field and the problem of phase transitions. (Russian) *Funkcional. Anal. i Priloen.* **2** (1968), no. 4, 44–57.

[6] Dobrushin, R. L. Description of a random field by means of conditional probabilities and conditions for its regularity. (Russian) *Teor. Verojatnost. i Primenen* **13** (1968), 201–229.

[7] Doukhan, P. *Mixing. Properties and examples.* Lecture Notes in Statistics, 85. Springer-Verlag, New York, 1994.

[8] Fernández, R. and Maillard, G. Construction of a specification from its singleton part. *ALEA Lat. Am. J. Probab. Math. Stat.* **2** (2006), 297–315.

[9] Geman, S. and Geman, D. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Trans. Pattern Anal. Mach. Intell*, **6** (1984), 721-741.

[10] Nahapetian, B. *Limit theorems and some applications in statistical physics.* Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], 123. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1991.

[11] Shental, N., Zomet, A., Hertz, T., and Weiss, Y. Learning and Inferring Image Segmentations using the GBP Typical Cut Algorithm. *In Proceedings of ICCV*, Volume 2 (October 13 - 16, 2003).