Homework Assignment 2

Compressed sensing and restricted isometry

Let us consider the model :

$$\boldsymbol{Y} = \boldsymbol{\Phi}\boldsymbol{\beta}^* + \boldsymbol{\xi}$$

where $\boldsymbol{Y} \in \mathbf{R}^n$, $\boldsymbol{\beta}^* \in \mathbf{R}^p$, $\boldsymbol{\Phi}$ is a known $n \times p$ matrix and $\boldsymbol{\xi} \in \mathbf{R}^n$ is the noise term such that $\|\boldsymbol{\xi}\|_2 < \varepsilon\sqrt{n}$, for some known parameter $\varepsilon \geq 0$. We say that $\boldsymbol{\Phi}$ satisfies the restricted isometry property of order s, RI(s), if there exists $\delta_s \in (0,1)$ such that for every s-sparse vector $\boldsymbol{\beta}$,

$$(1-\delta_s)\|\boldsymbol{\beta}\|_2^2 \leq \frac{1}{n}\|\boldsymbol{\Phi}\boldsymbol{\beta}\|_2^2 \leq (1+\delta_s)\|\boldsymbol{\beta}\|_2^2. \tag{1}$$

We consider the solution $\widetilde{\boldsymbol{\beta}}$ of the following minimization problem :

$$\min_{\boldsymbol{\beta}\in\mathbf{R}^p}\|\boldsymbol{\beta}\|_1 \quad \text{subject to} \quad \|\boldsymbol{Y}-\boldsymbol{\Phi}\boldsymbol{\beta}\|_2 \leq \varepsilon\sqrt{n}. \tag{2}$$

The goal is to prove the following result.

Theorem 1 *If $\boldsymbol{\beta}^*$ is s-sparse (that is $\|\boldsymbol{\beta}^*\|_0 \leq s$) and the matrix $\boldsymbol{\Phi}$ satisfies RI($2s$) property with $\delta_{2s} < \sqrt{2}-1$, then for some constant $c > 0$,*

$$\|\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\|_2 \leq c\varepsilon.$$

In particular, if $\varepsilon = 0$, we get exact reconstruction : $\widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta}^*$.

1. Prove that $\widetilde{\boldsymbol{\beta}}$ exists, that is problem (2) admits a solution.
2. Prove that under the conditions of the theorem, the model $\boldsymbol{Y} = \boldsymbol{\Phi}\boldsymbol{\beta}^*$ is identifiable.
3. We call the support of a vector $\boldsymbol{\beta}$ the set $S = \{j : \beta_j \neq 0\}$. Show that for any pair of vectors $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ with disjoint supports and of sparsity s and s', respectively,

   $$\frac{1}{n}|\langle \boldsymbol{\Phi}\boldsymbol{\beta}, \boldsymbol{\Phi}\boldsymbol{\beta}'\rangle| \leq \delta_{s+s'}\|\boldsymbol{\beta}\|_2\|\boldsymbol{\beta}'\|_2,$$

   where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbf{R}^p$.
4. Show that $\frac{1}{\sqrt{n}}\|\boldsymbol{\Phi}(\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^*)\|_2 \leq 2\varepsilon$.
5. Prove that $\|\widetilde{\boldsymbol{\beta}}\|_1 \leq \|\boldsymbol{\beta}^*\|_1$.
6. We introduce the vector $h = \boldsymbol{\beta}^* - \widetilde{\boldsymbol{\beta}}$ and decompose it into the sum of the vectors $h_{T_0}, h_{T_1}, h_{T_2}, \ldots$, each of sparsity at most s, with T_0 the set of indices corresponding to s nonzero entries of $\boldsymbol{\beta}^*$, T_1 the set of indices corresponding to s largest (in absolute values) entries [1] of $h_{T_0^c}$, T_2 is the set of indices corresponding to s largest entries of h which are neither in T_0 nor in T_1, etc.

   Using the triangle inequality, deduce from the previous question that

   $$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1.$$

1. We denote by T^c the complementary set of T.

7. Show that for $j \geq 2$,
$$\|h_{T_j}\|_2 \leq s^{-1/2} \|h_{T_{j-1}}\|_1.$$
Deduce from the last inequality that
$$\|h_{(T_0 \cup T_1)^c}\|_2 \leq s^{-1/2} \|h_{T_0^c}\|_1.$$

8. Combining the last two questions, show that
$$\|h_{(T_0 \cup T_1)^c}\|_2 \leq \|h_{T_0}\|_2.$$

9. Show that
$$\frac{1}{n} |\langle \mathbf{\Phi} h_{T_0 \cup T_1}, \mathbf{\Phi} h \rangle| \leq \frac{1}{n} \|\mathbf{\Phi} h_{T_0 \cup T_1}\|_2 \|\mathbf{\Phi} h\|_2 \leq 2\varepsilon \sqrt{1 + \delta_{2s}} \|h_{T_0 \cup T_1}\|_2$$
and that
$$(1 - \delta_{2s}) \|h_{T_0 \cup T_1}\|_2^2 \leq \frac{1}{n} \|\mathbf{\Phi} h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \big(2\varepsilon \sqrt{1 + \delta_{2s}} + \sqrt{2} \delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_2 \big).$$

10. Check that
$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha \varepsilon + \rho \|h_{T_0 \cup T_1}\|_2,$$
for two constants α and ρ to be specified.

11. Conclude.