

# PROJECTION ONTO A SHAPE MANIFOLD FOR IMAGE SEGMENTATION WITH PRIOR

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## ABSTRACT

Image segmentation with shape priors has received a lot of attention over the past years. Most existing work focuses on a linearized shape space with small deformation modes around a mean shape, which is relevant only when considering similar shapes. In this paper, we introduce a new framework that can handle more general shape priors. We model a category of shapes as a finite dimensional manifold, *the shape prior manifold*, which we approximate from the shape samples using dimensionality reduction techniques suchlike Laplacian eigenmaps. Unfortunately, this model does not provide an explicit projection operator onto the manifold. Our contribution is twofold. First, we calculate the low dimensional representation of any point not in the training set. Second, we properly define a projection operator onto the manifold by interpolating between shape samples using local weighted means. We show results both on synthetic and real shapes and demonstrate the potential of our method for segmentation tasks.

**Index Terms**— Shape manifold, graph Laplacian, segmentation, prior, dimensionality reduction

## 1. INTRODUCTION

### 1.1. Motivation

Image segmentation is an ill-posed problem due to various perturbing factors such as noise, occlusions, missing parts, cluttered data, etc. When dealing with complex images, some prior shape knowledge may be necessary to disambiguate the segmentation process. The use of such prior information in the deformable models framework has long been limited to a smoothness assumption or to simple parametric families of shapes. But a recent and important trend in this domain is the development of deformable models integrating more elaborate prior shape information.

An important work in this direction is the *active shape model* of Cootes et al. [1], but is limited due to a parametrized representation of shapes. Among the most recent works, Leventon, Grimson and Faugeras [2] compute parameterization-independent shape statistics, within the level set representation [3, 4], based on PCA on the signed distance functions of the training shapes. Several improvements to this approach

have been proposed in [5, 6]. Let us also mention a neat Bayesian prior shape formulation, based on a B-spline representation, proposed by Cremers et al. in [7].

Performing PCA on distance functions might be problematic since they do not define a vector space. To cope with this, Charpiat, Faugeras and Keriven [8] proposed shape statistics based on differentiable approximations of the Hausdorff distance. Similar ideas are presented in [9]. However, their work is limited to a linearized shape space with small deformation modes around a mean shape. Such an approach is relevant only when the learning set is composed of very similar shapes.

### 1.2. Contributions

In this paper, we introduce a new framework that can handle more general shape priors. We model a category of shapes as a smooth finite-dimensional submanifold of the infinite-dimensional shape space. In the sequel, we term this finite-dimensional manifold the *shape prior manifold*. This manifold cannot be represented explicitly. We approximate it from a collection of shape samples using a recent manifold learning technique called Laplacian eigenmaps [10] that constructs an embedding from data. This technique has been very recently applied in [11] to sets of shapes but it has never been used in the context of image segmentation with shape priors.

Our main contribution is to properly define the projection of a shape onto the *shape prior manifold*, by estimating the embedding with a regression function in its entire space and by interpolating between some carefully selected shape samples using local weighted means shapes.

The remainder of this paper is organized as follows. Section 2 is dedicated to learning the shape prior manifold from a finite set of shape samples using the Laplacian eigenmaps technique. Section 3 presents a method for the projection of a new shape on the embedding that enables to find the projection onto the *shape manifold*. In Section 4, we report on some numerical experiments which yield promising results with synthetic and real shapes.

## 2. LEARNING THE SHAPE PRIOR MANIFOLD

### 2.1. Definitions

In the sequel, we define a *shape* as a simple (i.e. non-intersecting) closed curve, and we denote by  $\mathcal{S}$  the space of such

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shapes. Please note that, although this paper only deals with 2 dimensional shapes, all ideas and results seamlessly extend to higher dimensions.

The space  $\mathcal{S}$  is infinite-dimensional. We make the assumption that a category of shapes, i.e. the set of shapes that can be identified with a common concept or object, e.g. fish shapes, can be modeled as a finite-dimensional manifold.

In the context of estimating the shape of an object in a known category from noisy and/or incomplete data, we call this manifold the *shape prior manifold*. In practice, we only have access to a discrete and finite set of example shapes in this category. We will assume that this set constitutes a "good" sampling of the *shape prior manifold*, where "good" stands for "exhaustive" and "sufficiently dense" in a sense that will be clarified below.

Many different definitions of the distance between two shapes have been proposed in the computer vision literature but there is no agreement on the right way of measuring shape similarity. The definition used in experiments presented in this paper are based on the representation of a curve in the plane by its signed distance function. In this context, the distance between two shapes can be defined as the Sobolev  $W^{1,2}$ -norm of the difference between their signed distance functions. Let us recall that  $W^{1,2}(\Omega)$  is the space of square integrable functions over  $\Omega$  with square integrable derivatives [8]:

$$d_{W^{1,2}}(S_1, S_2)^2 = \|\bar{D}_{S_1} - \bar{D}_{S_2}\|_{L^2(\Omega, \mathbb{R})}^2 + \|\nabla \bar{D}_{S_1} - \nabla \bar{D}_{S_2}\|_{L^2(\Omega, \mathbb{R}^n)}^2$$

where  $\bar{D}_{S_i}$  denotes the signed distance function of shape  $S_i$  ( $i = 1, 2$ ), and  $\nabla \bar{D}_{S_i}$  its gradient.

The method presented in this paper is not limited to this distance and other distance may be used, such as the symmetric difference between the region bounded by the two shapes or the Hausdorff distance [12, 8].

## 2.2. Manifold learning

Once some distance  $d$  between shapes has been chosen, classical manifold learning techniques can be applied, by building a neighborhood graph of the learning set of shape examples. Let  $(S_i)_{i \in 1, \dots, p}$  denote the  $n$  shapes of the learning set. An adjacency matrix  $(W_{i,j})_{i,j \in 1, \dots, p}$  is then designed, the coefficients of which measure the strength of the different edges in the neighborhood graph. See [10] for details.

Once a neighborhood graph is constructed from a given set of samples, manifold learning consists in mapping data points into a lower dimensional space while preserving the local properties of the adjacency graph. This dimensionality reduction with minimal local distortion can be achieved using spectral methods, i.e. through an analysis of the eigenstructure of some matrices derived from the adjacency matrix. Dimensionality reduction has enjoyed renewed interest over

the past years. Among the most recent and popular techniques are Isomap [13], the Locally Linear Embedding (LLE) [14], Laplacian eigenmaps [10], Diffusion maps [15]

Below, we present the mathematical formulation of Laplacian eigenmaps for data living in  $\mathbb{R}^n$ . An extension to shape manifolds is straightforward.

Let  $\mathcal{M}$  be a manifold of dimension  $m$  lying in  $\mathbb{R}^n$  ( $m \ll n$ ). For the time being, we take  $m = 1$  since generalization to any dimension  $m < n$  is immediate. The dimensionality reduction problem consists in finding a mapping (called an embedding)  $f : \mathcal{M} \rightarrow \mathbb{R}$  such that if two points  $x$  and  $z$  are close in  $\mathcal{M}$ , so are  $f(x)$  and  $f(z)$ . To characterize such an optimal mapping, the following inequality is stated in [10]:

$$|f(z) - f(x)| \leq d_{\mathcal{M}}(x, z) \|\nabla f(x)\| + o(d_{\mathcal{M}}(x, z)) \quad (1)$$

where  $d_{\mathcal{M}}$  is the geodesic distance on the manifold  $\mathcal{M}$ . The optimality condition then writes:

$$f^* = \arg \min_{f: \|f\|_{L^2(\mathcal{M})}=1} \int_{\mathcal{M}} \|\nabla f\|^2 \quad (2)$$

$$= \arg \min_{f: \|f\|_{L^2(\mathcal{M})}=1} \int_{\mathcal{M}} \mathcal{L}(f) f \quad (3)$$

$\mathcal{L} = -\text{div}(\nabla f)$  is the Laplace-Beltrami operator and the equivalence between (2) and (3) is due to the Stokes theorem. The sign  $*$  denotes the solution of a minimization problem and henceforth it will be used in that sense.

Solving the minimization problem (3) is equivalent to solving the eigen problem  $\mathcal{L}(f) = \lambda f$ . The optimal mapping is then given by the eigen functions corresponding to the  $m$  smallest non-zero eigenvalues of  $\mathcal{L}(f)$ , where  $m$  is the target dimension. Note that the latter dimension can either be known *a priori* or be inferred from the profile of the eigen spectrum.

In practice, a discrete counterpart to this continuous formulation must be used. Let  $x_1 \cdots x_p \in \mathbb{R}^n$  be  $p$  sample points of the  $m$  dimensional manifold  $\mathcal{M}$ . The corresponding neighborhood graph is denoted  $\mathcal{G}$  and its adjacency matrix  $(W_{i,j})_{i,j \in 1, \dots, p}$  is given by  $W_{i,j} = k(x_i, x_j)$ , where  $k(\cdot, \cdot)$  is mostly chosen to be the well known gaussian kernel  $k(x_i, x_j) = e^{-\frac{d(x_i, x_j)^2}{2\sigma^2}}$ .

Let also  $L = D - W$  where  $D$  is the diagonal matrix defined by  $D_{i,i} = \sum_j W_{i,j} = \sum_j k(x_i, x_j)$ . Matrix  $L$  represents the Laplacian of the graph  $\mathcal{G}$ . It is known to be a convergent approximation of the Laplace-Beltrami operator over the manifold  $\mathcal{M}$ . Note that the convergence is achieved under certain conditions depending on the probability density used to sample the manifold  $\mathcal{M}$ . This study is however out the scope of this article. For the sake of clarity, in this paper we outline the basic formulation for uniform probability density leading to the unnormalized graph Laplacian  $L$ . The discrete formulation of equation 3 [10] is

$$y^* = \arg \min_{y: y^T D y = 1} (y^T L y) \quad (4)$$

where  $y \in \mathbb{R}^p$ .  $y(i)$  denotes the  $i^{\text{th}}$  coordinate of vector  $y$  and is the 1-dimensional embedding value of the point  $x_i$ . Note that  $y(i)$  is equivalent to  $f(x_i)$  in the continuous formulation. Solving equation 4 consists merely of solving the general eigenproblem  $Ly = \lambda Dy$  and keeping the eigen vector corresponding to the smallest non-zero eigenvalue. When the optimal embedding is of dimension  $m > 1$ , the energy involved is then generalized into:

$$Y^* = \arg \min_{Y: Y^T D Y = I} \text{Tr}(Y^T L Y) \quad (5)$$

where  $Y = [y_1, \dots, y_m]$  is a  $(p \times m)$  matrix. ( $y_1$  is equivalent to  $y$  in equation 4). For any  $i = 1, \dots, m$ , the  $i^{\text{th}}$  row vector  $y(x_i) = y(i) = [y_1(i), \dots, y_m(i)] \in \mathbb{R}^p$  of matrix  $Y$  represents the  $m$  dimensional embedding of the point  $x_i \in \mathbb{R}^n$ . Such notations will be used from now on. Optimal dimensionality reduction is achieved by finding the eigenvectors  $y_1, \dots, y_m$  of matrix  $L$  corresponding to the  $m$  smallest non-zero eigenvalues. Laplacian eigenmaps for shapes  $(S_i)_{i=1, \dots, p}$  is computed by using the same procedure (fig. 2)

Although the Laplacian eigenmaps technique is a powerful tool for dimensionality reduction, it does not give access to neither an explicit projection onto the manifold nor its embedding. We can thus identify two major limitations: First, the embedding values calculated by the solution of equation 5 is restricted to the training samples. Computing the embedding of points not in the training set is known as the *out of sample problem*. Second, the *preimage problem* consists in estimating a shape on the *shape prior manifold* given an embedding value. Note that we also need to describe the *shape prior manifold* in between the training shapes.

### 3. PROJECTION ONTO A SHAPE MANIFOLD

We aim this section at **1.** computing the embedding of a new data point  $x_{p+1}$ ,  $x_{p+1} \neq x_i \quad \forall i = 1, \dots, p$  **2.** retrieving the corresponding shape associated to such embedding value.

#### 3.1. Out of sample problem

In this part, we tackle the first limitation presented in the previous section and show that it can be solved by means of a regression function of the discrete embedding. The most similar approach known in the literature to relies on the Nystrom extension [16]: it consists in extending the eigenvector of a discrete operator to all the space. In this work, we take a different approach leading to a solution expressed as a regularization function of the discrete embedding.

We start again with the formulation for data living  $\mathbb{R}^n$ . The embedding of the new point  $x_{p+1}$  requires matching some properties. First, it should use the discrete embedding previously computed from the training samples. Indeed, computing a new embedding with the samples  $(x_i)$ ,  $\forall i = 1, \dots, p+1$  is not relevant and above all would not be efficient. Then, the

point  $x_{p+1}$  may not belongs to the manifold  $\mathcal{M}$ , Now, we reformulate equation 5 with  $n + 1$  points. Let  $w = (w_i)_{i=1, \dots, p}$  be defined by  $w_i = k(x_{p+1}, x_i)$  and  $L_n$  such that

$$L_n = \begin{bmatrix} L_o & -w \\ -w & \sum_{i=1}^p w_i \end{bmatrix} \quad (6)$$

where  $L_o$  may be the Laplacian matrix obtained with the points  $x_1, \dots, x_p$  or an updated version depending on  $w$ . Whatever the choice, we will show in the following lines that it does not influence the final result. Following equation 5, the unconstrained energy to minimize can then be written

$$\min_{Z: Z^T D Z = I} \text{Tr}(Z^T L_n Z)$$

where  $Z = [z(1)^T, \dots, z(p+1)^T]^T$  is a  $(p+1 \times m)$  matrix. Since the embedding of the  $p$  points  $x_1, \dots, x_p$  is already known, we add the constraints  $\forall i = 1, \dots, p \quad z(i) = y^*(i)$  from the solution of equation 5 and obtain:

$$z^*(p+1) = \arg \min_{z(p+1): Z=[Y^{*T} z(p+1)^T]^T} \text{Tr}(Z^T L_n Z) \quad (7)$$

Deriving equation 7 leads to the mapping  $\hat{z}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ :

$$\hat{z}(x) = \frac{\sum_i k(x, x_i) y^*(i)}{\sum_i k(x, x_i)} \quad (8)$$

The result pointed up in equation 8 is of particular interest since it does not depend on  $L_o$  and the solution is expressed by means of the Nadaraya-Watson kernel widely used in the statistical learning literature. The function  $\hat{z}(x)$  can be seen as a regression function estimating the continuous embedding. Note that  $\hat{z}(x_i) \neq y^*(x_i)$ ,  $\forall i = 1, \dots, p$ , so we have to consider the values  $\hat{y}(x_i) = \hat{z}(x_i)$  instead of  $y^*(x_i)$ .

We applied this projection to data sets of shapes  $(S_i)_{i=1, \dots, p}$  instead of euclidian points  $(x_i)_{i=1, \dots, p}$ . The results obtained are illustrated in figures 1 and 3

#### 3.2. Finding the corresponding point in the shape space

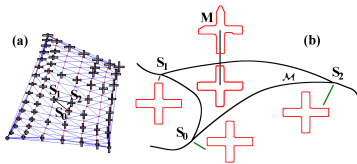
Let  $P_{\mathcal{M}}(S) \in \mathcal{S}$  be the projection onto the *shape prior manifold*  $\mathcal{M}$ . Once the embedding  $\hat{z}(S)$  of a new shape point  $S \in \mathcal{S}$  has been computed, the shape  $P_{\mathcal{M}}(S) \in \mathcal{S}$  has to be found. From now on, we suppose the dimension  $m$  of the *shape manifold* to be fixed. We basically assume the shape  $P_{\mathcal{M}}(S)$  to be a *weighted mean shape* that interpolates between  $m + 1$  samples of a *neighborhood system*  $\mathcal{N} = (S_0, \dots, S_m)$ .  $\mathcal{N}$  is determined based on a  $m$  dimensional Delaunay triangulation in the reduced space of the data  $\hat{y}(j)$ ,  $\forall j \in 1, \dots, p$  (fig. 1) Indeed,  $\mathcal{N}$  corresponds to the points of the  $m$  dimensional triangle in which the point  $\hat{z}(x)$  falls. The barycentric coefficients can be immediately computed:  $\Lambda = (\lambda_0, \dots, \lambda_m)$  with  $(\lambda_i \geq 0, \sum \lambda_i = 1)$  Thus, the local interpolation of the shape manifold is given by:

$$\bar{S}_{\mathcal{N}}(\Lambda) = \arg \min_S \sum_{i=0}^m \lambda_i d(S_i, S)^2$$

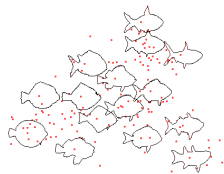
#### 4. APPLICATION TO SEGMENTATION WITH SHAPE PRIOR & CONCLUSION

We propose to apply the method presented in this paper in the context of image segmentation with shape priors. Without loss of generality, the method is stated as a variational problem attempting to minimize an energy  $E^T(S) = E^{ac}(S) + \alpha E_{\mathcal{N},\Lambda}^p(S)$  (very basic formulation).  $E_a(S)$  is the common energy used in the active contour framework.  $E_{\mathcal{N},\Lambda}^p(S) = \sum_{i=0}^m \lambda_i d(S_i, S)^2$  is the prior term that attracts the evolving shape towards the *shape prior manifold*, for a given neighborhood system  $\mathcal{N}$  and barycentric coefficients  $\Lambda$ .  $\alpha$  is a parameter that influences the importance of the prior term. The energy  $E^T(S)$  is minimized by using calculus of variations. Results are presented in figure 3.

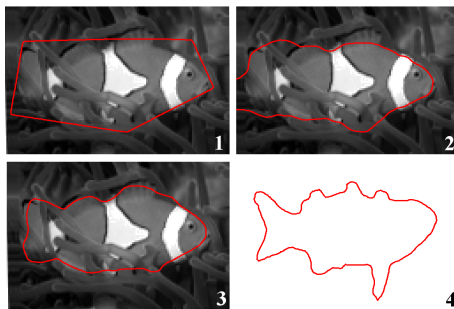
In this paper, we presented a new technique that handles general shape prior and the results obtained show the potential of the method.



**Fig. 1.** **a:** 2-dimensional representation of 150 crosses and its Delaunay triangulation. **b:** Projection of a corrupted shape on the *shape prior manifold*



**Fig. 2.** 2-dimensional representation of 150 fishes [SQUID database]



**Fig. 3.** Fish segmentation **1:** initial contour **2:** active contour without shape prior **3:** active contour with shape prior **4:** reprojection of the final result on the *shape manifold*

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