On projective plane curve evolution

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Abstract

In this paper, we investigate the evolution of curves of the projective plane according to a family of projective invariant intrinsic equations. This is motivated by previous work for the Euclidean [11, 12, 14] and the affine cases [21, 22, 3, 2] as well as by applications in the perception of two-dimensional shapes. We establish the evolution laws for the projective arclength and curvature. Among this family of equations, we define a "projective heat equation" [7] and establish the link with the projective evolution of curves in $\mathbb{R}^2$.

Keywords: multi-scale analysis, partial differential equations, projective geometry

1 Introduction

The use of partial differential equations and curve or surface evolution theory in image analysis became a major research topic in the past years (see [18]) leading to applications in image de-noising and de-blurring [19], in selective smoothing and edge detection [1, 17], in contrast enhancement [20], in shape segmentation [5]. Recently, applications were found in problems usually addressed by the computer vision community: intrinsic flows [14, 21] hold very good geometric smoothing properties and allow the computation of local differential invariants [9]. Motivated by the importance of projective geometry in computer vision, we found it natural to extend the Euclidean [14] and affine [21] cases to the projective one.

2 Geometric flows

Let $\mathcal{L}$ be a Lie group operating on some objects. A quantity $q$ depending on these objects is called an invariant of $\mathcal{L}$ if, whenever a transformation $L \in \mathcal{L}$ changes $q$ into $q'$, we have $q' = \alpha(L)q$, where $\alpha$ is a function of $L$ alone, i.e. does not depend on the object which is transformed. If $\alpha \equiv 1$, then $q$ is called an absolute invariant.

Differential invariants are special invariants based on local transformations (see [13]).

Let $C : \mathbb{R} \rightarrow \mathbb{R}^2$ be a plane curve of parameter $p$. The first and the second differential invariants for the Euclidean group $\{m \mapsto Rm + T \mid R \text{ rotation, } T \text{ translation}\}$ are
the well known Euclidean arclength $v$ and curvature $\kappa$ defined by:

\[
\begin{cases}
\frac{\partial v}{\partial s} &= \| \frac{\partial \mathbf{r}}{\partial s} \| \\
\kappa &= \| \frac{\partial^2 \mathbf{r}}{\partial s^2} \|
\end{cases}
\]

(1)

which are preserved by rotations and translations.

The corresponding invariants for the group of proper affine motions \{ $m \mapsto Am + B \mid [A] > 0, B \in \mathbb{R}^2$ \} are the affine arclength $s$ and curvature $\mu$ defined by:

\[
\begin{cases}
\frac{\partial s}{\partial p} &= \frac{\alpha C(p, t)}{\alpha C(p, 0)}^{1/3} \\
\mu &= \frac{\alpha C(p, t)}{\alpha C(p, 0)}
\end{cases}
\]

(2)

which are invariants for affine proper motions, and absolute invariants for special affine motions ( \{ $m \mapsto Am + B \mid [A] = 1, B \in \mathbb{R}^2$ \} ).

Circles (and straight lines) are the only curves with constant Euclidean curvature.

In the affine case, constant affine curvature is obtained for the conics ($\mu = 0$ for a parabola, $\mu > 0$ for an ellipse and $\mu < 0$ for an hyperbola).

Given an initial plane curve $C_0(p) : \mathbb{R} \to \mathbb{R}^2$, the associated geometric flow (see [16]) is the family of curves $C(p, t) : \mathbb{R} \times [0, \tau) \to \mathbb{R}^2$ evolving according to the following law:

\[
\begin{cases}
\frac{\partial C(p, t)}{\partial t} &= \frac{\alpha C(p, t)}{\alpha C(p, 0)} \\
\frac{\partial C(p, 0)}{\partial p} &= C_0(p)
\end{cases}
\]

(3)

where $r$ is the group arclength ($v$ for the Euclidean geometric flow, $s$ for the affine one). Contrary to the classical heat flow $G_t = G_{pp}$, these flows are intrinsic (i.e. don’t depend on the parameterization $p$ of the initial curve). They are invariant for the considered Lie group. Their “smoothing” properties may be summarized as follow ([14, 21]): closed curves evolve toward a convex one and then disappear shrinking toward a circle point (Euclidean case) or an ellipse point (affine case).

For a given group, a plane curve is defined up to a group transformation by its group arclength and curvature. Hence, it is natural to study these flows through the evolution of the arclength and curvature. With $g_a = \frac{\partial g}{\partial p} = \frac{\partial g}{\partial p}$, we have:

\[
\begin{cases}
\frac{\partial g_a}{\partial s} &= -g_s \kappa^2 \\
\frac{\partial g_a}{\partial t} &= -\kappa^3 - \frac{\partial^2 g_a}{\partial s^2} \\
\frac{\partial g_a}{\partial t} &= -2g_s \mu / 3 \\
\frac{\partial g_a}{\partial s} &= \frac{4}{3} \mu^3 + \frac{1}{3} \frac{\partial^3 g_a}{\partial s^3}
\end{cases}
\]

(4)

3 Projective geometry

Like in equations (1) and (2), it is possible to define the projective arclength and curvature of a plane curve in $\mathbb{R}^2$. However, this leads to too complex expressions. The idea is to embed such a curve in the real projective plane $\mathbb{P}^e$. One can see $\mathbb{P}^e$ as the set of the lines of $\mathbb{R}^3$ going through the origin. An element of $\mathbb{P}^e$ is represented by its homogeneous coordinates $(x, y, z)$ where $(x, y, z)$ and $(\lambda x, \lambda y, \lambda z), (\lambda \neq 0)$ are different coordinate vectors of the same projective point.

Let $B(p) : \mathbb{R} \to \mathbb{P}^e$ be a smooth curve of the projective plane. Using standard results of projective differential geometry [4], we change $B(p)$ by a scale factor $\lambda(p)$ and
characterize its projective arclength $\sigma$ and curvature $k$ introducing the Cartan point $A = \lambda B$, and the Cartan frame $(A, A^{(1)}, A^{(2)})$ which satisfy the projective Frenet equations:

$$\frac{dA}{d\sigma} = A^{(1)}$$
$$\frac{dA^{(1)}}{d\sigma} = -kA + A^{(2)}$$
$$\frac{dA^{(2)}}{d\sigma} = -A - kA^{(1)}$$

and the condition:

$$|AA^{(1)}A^{(2)}| = 1$$

Note that $B$ and $A$ are different coordinate vectors of the same projective point. The point $A^{(1)}$ is on the tangent to the curve in $A$ and the line $\langle A, A^{(2)} \rangle$ is the projective normal. Functions $k$ and $\sigma$ are invariant under the action of the projective group and characterize the curve up to a projective transformation.

The plane curves with a constant projective curvature are (see [10]):

- If $k = k_0 = -3/32^{1/3}$: the exponential $(y = e^x)$
- If $k < k_0$: the general parabola $(y = x^m, m \notin \{2, \frac{1}{2}, -1\})$
- If $k > k_0$: the logarithmic spiral $(\rho = e^{m\theta}, m \neq 0)$

4 Projective invariant intrinsic flows

The law $A_t = A_{\sigma \sigma}$ investigated in [7] could be thought as a natural extension of the Euclidean and affine cases. Yet, this law raises some contradictions. For instance, according to the expression of $k_t$ in [7], curves with a constant initial curvature should evolve keeping a constant curvature. Actually, it’s not the case (see [10]).

The reason why it is so is that the Cartan point $A(p, t)$ is some particular representant for the projective point $B(p, t)$ and depends on the curve and its spatial derivatives at $(p, t)$. As a result, one can’t expect an arbitrary differential equation $\{A(p, 0) = A_0(p); A_t = f(p, t)\}$ to be such that $A(p, t)$ will still be the Cartan point of the curve at time $t > 0$.

This leads us to consider the evolution law

$$\begin{align*}
A(p, 0) &= A_0(p) \text{ (A_0 Cartan point of the initial curve)} \\
A_t(p, t) &= \alpha A + \beta A^{(1)} + \gamma A^{(2)}
\end{align*}$$

where $f(p, t)$ has been decomposed on the Cartan frame, and to find out which conditions on $(\alpha, \beta, \gamma)$ will assure that $A(p, t)$ remains the Cartan point.

Another way to see this is to consider the surface $S = \{A(\sqrt{\cdot} \cup)\}$ of $R^3$. The reason why this is a well-defined surface is because there is no scale factor on $A$ even though it represents a projective point of $P^6$. Now, in order for (7) to be a well-defined PDE on $S$, the vector $A_t$ has to belong to the tangent plane $T_S$ at $(p, t)$. The right hand
side contains the vector \( A^{(1)} \) which belongs to \( T_\mathcal{S} \) but the vector \( \alpha A + \gamma A^{(2)} \) does not in general belong to \( T_\mathcal{S} \) unless \( \alpha \) and \( \gamma \) are dependent. In fact the condition is even stronger since not only \( A_t \) must belong to \( T_\mathcal{S} \) but also, as stated above, \( \mathbf{A} \) must remain a Cartan point.

We get the following result (see [8] for the proof):

**Proposition 1** The differential equation (7) has a meaning (i.e. \( \mathbf{A}(p,t) \) is the Cartan point of the curve at time \( t \)) if and only if:

\[
\alpha = \frac{1}{3+k_\sigma} \left[ -\frac{1}{3}k_{\sigma^2} - \frac{3}{2}k_\sigma^2 \gamma_\sigma - k_\sigma \left( \frac{7}{3}k_\gamma + \frac{17}{6} \gamma_\sigma^2 + \beta_\sigma \right) - \frac{8}{3} k^2 \gamma_\sigma \\
+ k(\gamma - \frac{5}{3} \gamma_\sigma^2) + \gamma_\sigma^2/2 - \gamma_\sigma^3/6 \right] \tag{8}
\]

In this case, the projective arclength and curvature evolve according to:

\[
\frac{g_t}{g} = \alpha + \beta_\sigma - \frac{1}{3}(k_\gamma - \gamma_\sigma^2) \tag{9}
\]

\[
k_t = -\alpha_\sigma^2 + \frac{3}{2} \gamma_\sigma + \frac{\gamma_\sigma^3}{6} + k(\frac{2}{3} \gamma_\sigma^2 - 2\alpha) \\
+ k_\sigma (\beta + \frac{7}{6} \gamma_\sigma) + \frac{\gamma_\sigma^3}{3}(k_\sigma^2 + 2k^2) \tag{10}
\]

where \( g = \frac{d\sigma}{dt} \).

Note that \( A_t = A_{\sigma \sigma} \) is the case \((\alpha, \beta, \gamma) = (-k, 0, 1)\), thus doesn’t satisfy condition (8), hence the previous contradictions.

Moreover, if \( \beta \) and \( \gamma \) are projective invariant intrinsic quantities, then \( \alpha \) defined by equation (8) is a projective invariant intrinsic quantity too. Therefore, we get:

**Corollary 1** Let \( \beta \) and \( \gamma \) be some projective invariant intrinsic quantities, let \( \alpha \) be defined by equation (8). The differential equation (7) defines a projective invariant intrinsic flow. The evolution of the projective arclength and curvature of the curves is given by equations (9, 10).

## 5 The projective “heat flow”

Among all the possible choices for \((\beta, \gamma)\), it turns out that the simplest one \((0, 1)\) is also the right one for a projective “heat flow” extending the Euclidean and affine cases. Some intuitive justification could be:

- \( \beta A^{(1)} \) is on the tangent in \( \mathbf{A} \). Thus, the choice of \( \beta \) has no importance: changing \( \beta \) doesn’t modify the family of curves obtained but only their parameterization \( p \) (see [21]).

- \((\beta, \gamma) = (0, 1)\) are the components of \( A_{\sigma \sigma} \) on \((A^{(1)}, A^{(2)})\). The induced \( \alpha \) could be considered as a corrected component on \( \mathbf{A} \).

However, the deep reason for this choice is that it gives the same flow as \( \mathcal{C}_t = C_{\sigma \sigma} \) in \( \mathbb{R}^3 \) (see next section). Consequently, we have from proposition 1 the following statement:
Proposition 2 Let $\alpha$ be:

$$\alpha = \frac{1}{9 + 3k_\sigma}(3k - 7kk_\sigma - k_\sigma^2)$$

Let $B_0(p)$ be a curve of $P^3$ and $A_0(p)$ its Cartan points. We define its projective heat flow as the solution of:

$$\begin{align*}
A(p, 0) &= A_0(p) \\
A_t(p, t) &= \alpha A + A^{(2)}
\end{align*} \quad (11)$$

Let $g = \frac{dg}{dt}$. The projective arclength and curvature evolve according to:

$$\begin{align*}
g_t &= -\frac{1}{9 + 3k_\sigma}(8kk_\sigma + k_\sigma^2) \\
k_t &= \frac{2}{3}k^2 + \frac{1}{3}k_\sigma^2 - 2\alpha k - \alpha_\sigma^2
\end{align*} \quad (12, 13)$$

6 Going back to $\mathbb{R}^2$

We prove in [8] that:

Proposition 3 Given an initial curve in $P^3$, let $B_0(p)$ be any coordinate vector of it.

1. The flow defined by

$$\begin{align*}
B(p, 0) &= B_0(p) \\
B_t(p, t) &= B_{\sigma\sigma}
\end{align*} \quad (14)$$

is intrinsic and doesn’t depend on the choice of $B_0$ (i.e. $B_0(p)$ and $\phi(p)B_0(p)$ give the same family of curves).

2. This flow is the projective heat flow defined by equation (11) up to a parameterization of the curves.

3. Let $\lambda$ be the Cartan scale factor ($A = \lambda B$). ($\sigma, k, \lambda$) define $B$ up to a projective transformation. Their evolution is given by:

$$\begin{align*}
g_t &= -\frac{1}{9 + 3k_\sigma}(8kk_\sigma + k_\sigma^2 + 18\lambda_\sigma^2) \\
k_t &= \frac{2}{3}k^2 + \frac{1}{3}k_\sigma^2 - 2k_\sigma^2 - 2k_\sigma\lambda_\sigma \\
\lambda_t &= \frac{-1}{9 + 3k_\sigma}[k_\sigma^3 + 3k_\sigma(\lambda_\sigma^2 - 3\lambda_\sigma)] + 4k_\sigma - 3\lambda_\sigma + 9(\lambda_\sigma^2 - \lambda_\sigma^2)]
\end{align*} \quad (15)$$

where $g = \frac{d\sigma}{dp}$, $\lambda = \log|\lambda|$, $P = \lambda_\sigma^2 - \lambda_\sigma^2 - k + \lambda_t$

Let $C_0(p) = (x_0, y_0)$ be a real plane curve, it is then easy to prove that:

Corollary 2 The flow defined by $\{C(p, 0) = C_0 ; C_t = C_{\sigma\sigma}\}$ is a projective invariant flow. It gives the same family of curves through the map $\left(\frac{x}{y}, \frac{y}{x}\right)$ as the projective heat flow (11) with initial curve $(x_0, y_0, 1)$. Let $C(p, t) = (x, y)$ and $\lambda$ be the Cartan scale of $(x, y, 1)$, the evolution of the projective arclength and curvature of $C$ is given by equations (15).
This was already proved in [15], even though the argument in [16] about the relationship between different coordinate vectors is incorrect (see proposition 3 above)

7 Conclusion

We have established a link between the invariant projective flow defined in $\mathbb{R}^2$ [16, 15] and the one defined in $\mathcal{P}^\mathbb{E}$ [7]. We have defined the projective heat equation in three equivalent ways: $A_t = \alpha A + A^{(2)}$ ($\alpha$ given by equation (8)) or $B_t = B_{\sigma \sigma}$ in $\mathcal{P}^\mathbb{E}$, and $C_t = C_{\sigma \sigma}$ in $\mathbb{R}^2$. As expected, the connection is not trivial but simple enough. The advantage of the definition in $\mathcal{P}^\mathbb{E}$ [7] which we have modified here to make it entirely correct is that: a) it does not depend on the particular coordinates used to represent $\mathcal{P}^\mathbb{E}$ and b) it has allowed us to establish the evolution of the projective arclength and curvature. There remains to see if it is possible to define a projective scale-space as in the Euclidean and affine cases. Of particular interest would be to compare our approach with the one developed by Dibos [6].

References


